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Programming Under Uncertainty: The Complete Problem

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by

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ABSTRACT

We define the complete problem of a two-stage linear programming under uncertainty, to be:

$$\begin{aligned} \text{Minimize} \quad z(x) &= E_\xi\{c\,x + q^+y^+ + q^-y^-\} \\ \text{subject to} \quad & A\,x = b \\ & T\,x + I\,y^+ + I\,y^- = \xi \\ & x \geq 0 \quad y^+ \geq 0 \quad y^- \geq 0 \end{aligned}$$

where x is the first-stage decision variable, the pair (y^+, y^-) represents the second-stage decision variables. In order to solve this class of problem, we derive a convex programming problem, whose set of optimal solutions is identical to the set of optimal solutions of our original problem. This problem is called the <u>equivalent</u> convex programming. If the random variable ξ has a continuous distribution, we give an algorithm to solve the equivalent convex program. Moreover, we derive explicitly the equivalent convex program for a few common distributions.

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I. INTRODUCTION

The <u>standard form</u> for the two-stage linear program under uncertainty is:

$$(1) \qquad \text{Minimize} \quad z(x) = E_{\xi}\{cx + qy\}$$

$$\text{subject to} \qquad \begin{aligned} Ax \quad &= b \\ Tx + My &= \xi \qquad \xi \in (\Xi, \mathcal{B}, F) \\ x \geq 0 \quad y &\geq 0 \end{aligned}$$

where A is a matrix $m \times n$, T is $\bar{m} \times n$, M is $\bar{m} \times \bar{n}$, ξ is a random vector whose probability space is $(\Xi, \mathcal{B}, F)$. This problem (1) belongs to the class of stochastic linear programming problems for which one seeks a <u>here-and-now solution</u>. One interprets problem (1) as follows: the decision maker selects the activity levels for x, say $x = \hat{x}$, he then observes the random event $\xi = \hat{\xi}$ and he is finally allowed a corrective action y, such that $y \geq 0$, $My = \hat{\xi} - T\hat{x}$ and qy is minimum. This second stage decision y, is taken when no "uncertainties" are left in the problem.

The decision maker wants to minimize the sum of his fixed costs (cx) and of the penalty costs he may expect when he has selected given activity levels (x). It is clear from this interpretation that we could also write the objective function of (1)

$$(2) \qquad z(x) = cx + E_{\xi}\{\min qy \mid x\}.$$

All quantities considered here belong to the reals, denoted R. Vectors will belong to finite-dimensional real vector spaces R^n and whether they are to be regarded as row vectors or column vectors will always be clear from

the context in which they appear. Thus, for example, the expressions

$$\chi = (\chi_1, \ldots, \chi_i, \ldots, \chi_{\bar{m}})$$

$$Tx = \chi$$

$$y^+y^- = \sum_{i=1}^{\bar{m}} y_i^+ y_i^-$$

are easily understood. No special provisions will be made for transposing vectors.

The random vector $\xi = (\xi_1 \ldots \xi_i \ldots \xi_{\bar{m}})$ is a "numerical" random vector, i.e. $\Xi \subset R^{\bar{m}}$, $\mathfrak{B}$ is an algebra or a σ-algebra and F is a probability distribution function from which could be obtained a probability measure. $(\Xi_i, \mathfrak{B}_i, F_i)$ is the probability space of the random variable ξ_i. We only need independence of ξ and x: our first-stage decision has then no effect on $(\Xi, \mathfrak{B}, F)$.

If for every finite interval, $F_i(\xi_i)$ has a finite number of discontinuity points, then we can always integrate by parts $\int g_i(\xi_i) dF_i(\xi_i)$, where $g_i(\xi_i)$ is a linear function of ξ_i. If it exists, we denote the density function of ξ_i by $f_i(\xi_i)$ $i = 1, \ldots, \bar{m}$ and let α_i and β_i be respectively the greatest lower bound and least upper bound, if they exist, of ξ_i. We assume that $E_{\xi_i}\{\xi_i\}$ exists for all $i = 1, \ldots, \bar{m}$.

We say that problem (1) is <u>complete</u> when the matrix M (after an appropriate rearrangement of rows and columns) can be partitioned in two parts, whose first part is an identity matrix and the second part is the negative of an identity

matrix, $M = (I, -I)$.

The standard form of the problem to be studied in this article is then

$$
\begin{array}{llll}
(3) & \text{Minimize} & z(x) = E_\xi\{cx + q^+y^+ + q^-y^-\} & \\
 & \text{subject to} & Ax \qquad\qquad\quad = b & \\
 & & Tx + Iy^+ - Iy^- = \xi & \xi \epsilon (\Xi, \mathfrak{B}, F) \\
 & & x \geq 0, y^+ \geq 0, y^- \geq 0 &
\end{array}
$$

where we partitioned the vectors q and y of the standard form (1) in (q^+,q^-) and (y^+,y^-), respectively. The fact that $m = 0$ (i.e. there are no constraints of type $Ax = b$) does not alter the characteristics of our problem.

Among all classes of special cases of the two-stage linear programs under uncertainty, the "complete" case seems to cover the largest class of possible applications. One can think of the vector x as representing the activity levels of a production plant, constrained by $Ax = b$, $x \geq 0$. T is the "transformation" of these activity levels into sellable goods. $\chi = Tx$, is then the amount of goods the producer decides to place on a market where the demand, ξ, is only known in probability. y^+ and y^- represent the "errors" the producer made in estimating the demand; q^+ and q^- are penalty costs for making these "errors". For instance, an inventory type problem has $T = I$, q^+ represents the unit shortage cost, and q^- the unit holding cost, and $Ax = b$ the capacity, budget, technology,... constraints. It can be shown that the correlations between the ξ_i do not enter the problem; we do not need the independence of the ξ_i. We denote the marginal distribution functions by $F_i(\xi_i)$ $i = 1,\ldots\ldots,\bar{m}$.

The first section of this report shows the existence of an equivalent separable convex program to (3). In the second section we let the random variable ξ assume different distributions, and we derive the corresponding equivalent convex programs. Finally, we suggest an algorithm for solving (3) when ξ has a continuous distribution.

II. THE EQUIVALENT SEPARABLE CONVEX PROGRAM

We say that a programming is equivalent to another programming problem if their set of optimal solutions is identical. Let us consider

$$(4) \qquad \text{Minimize } z(x) = cx + Q(x)$$
$$\text{subject to } \quad Ax = b$$
$$x \geq 0$$

where

$$(5) \qquad Q(x) = E_{\xi \in (\Xi, \mathcal{B}, F)} \{Q(x,\xi)\}$$

and

$$(6) \qquad Q(x,\xi) = \{\text{Min } q^+y^+ + q^-y^- \mid y^+ - y^- = \xi - Tx,\ y^+ \geq 0,\ y^- \geq 0\}.$$

(7) Proposition: (4) is equivalent to (3).

By (5), definition of $Q(x)$ and (2), the objective functions of (3) and (4) are identical. It suffices to show that (3) and (4) have the same set of feasible solutions.

Since we seek a here-and-now solution, a solution to (3) is not a pair (x,y), but a vector x. Our decision y is taken when the random event has occurred.

Our second stage problem

$$(8) \qquad \text{Minimize } q^+y^+ + q^-y^-$$
$$\text{subject to } \quad Iy^+ - Iy^- = \xi - Tx$$
$$y^+ \geq 0 \quad y^- \geq 0$$

is always feasible, whatever be the values assumed by ξ and x; it is always possible to express any number as the difference of two non-negative numbers. The constraints limiting the here-and-now decision are: $Ax = b$, $x \geq 0$, i.e. (3) and (4) have the same set of feasible solutions. If (3) is (in)feasible so is (4) and vice versa.

The word <u>complete</u>, which was used to define the class of linear programs under uncertainty of the form (3), can now be justified by the properties of the solution set, viz.: every x satisfying the "fixed" constraints: $(Ax = b, x \geq 0)$ is automatically a feasible solution to problem (3). This is not the case in general for linear programs under uncertainty.

Let

$$K = \{x \mid Ax = b,\ x \geq 0\}.$$

If $K = \emptyset$ we define $\min_{x \in K} z(x) = -\infty$.

(10) <u>Proposition</u>: (4) is a convex program.

Since K is a convex set and cx is a linear function of x, it suffices to show that $Q(x)$ is convex in x. It is easy to verify that $Q(x,\xi)$ is convex in x (see (6)). The operator E_ξ applied to $Q(x,\xi)$, $\xi \in \Xi$, forms a positive weighted linear combination of convex functions in x. The resulting function $Q(x)$ is thus convex.

In what follows, we assume that (3) is solvable, i.e. $z(x)$ attains its minimum on K. We also assume that K has a non-empty interior. We now show that the Equivalent Convex Programming problem (4) is a Separable Convex Programming Problem [2, p. 482] and this, contrary to the assertion found in the Appendix to [4, p. 216].

The second part of this section describes some useful characteristics of the objective function of (4). The last part is devoted to show how the existing solution methods for separable convex programs could be used.

A. <u>$Q(\chi)$ is separable</u>.

Let

$$\chi_i = T_i x \quad \text{where } T_i \text{ is the } i^{th} \text{ row of } T$$

and

$$Q(\chi) = \quad x) \quad \text{when } \chi = Tx .$$

None the less, we should not confuse $Q(\chi)$ and $Q(x)$. Their domains being subsets of $R^{\bar{m}}$ and R^n, respectively.

If the function $Q(\chi)$ can be written in the form

$$Q(\chi) = \sum_{i=1}^{\bar{m}} Q_i(\chi_i)$$

where

$$Q_i(\chi_i) \text{ is a convex function}$$

and

$$\chi = (\chi_1,\dots\dots,\chi_{\bar{m}})$$

then $Q(\chi)$ is called <u>convex-separable</u>.

For a selected x (i.e. χ) and given ξ, the problem to be solved in the second stage is:

$$(11)\qquad P(\chi,\xi) = \text{Minimum} \sum_{i=1}^{\bar{m}} q_i^+ y_i^+ + \sum_{i=1}^{\bar{m}} q_i^- y_i^-$$

$$\text{subject to}\quad y_i^+ - y_i^- = \xi_i - \chi_i \qquad i = 1,\ldots,\bar{m}$$

$$y_i^+ \geq 0 \qquad y_i^- \geq 0$$

The dual to the linear program (11) is:

$$(12)\qquad Q(\chi,\xi) = \text{Maximum} \sum_{i=1}^{\bar{m}} \pi_i(\xi_i,\chi_i)(\xi_i - \chi_i)\ .$$

$$\text{subject to}\quad -q_i^- \leq \pi_i(\xi_i,\chi_i) \leq q_i^+ \qquad i = 1,\ldots,\bar{m}$$

We have already seen that for any given pair (χ, ξ), problem (11) is always feasible; problem (12) is feasible iff $\forall i$ the interval $[-q_i^-, q_i^+] \neq \emptyset$. These last conditions are completely independent of the values assumed by χ and ξ. Using the Existence Theorem (duality theory in linear programming), we establish the following:

(13) <u>Proposition</u>: (11) is solvable iff $q^+ + q^- = \tilde{q} \geq 0$.

The permanent $(\forall \chi, \forall \xi)$ feasibility of (11) and the proposition we just established implies that if the assumption $q^+ + q^- \geq 0$ was not satisfied, then

$$P(\chi, \xi) = -\infty \qquad \forall \xi,\ \forall \chi \quad (\forall x)$$

$$E_\xi\{P(\chi, \xi) = -\infty \qquad \forall \chi$$

and

$$z(x) = -\infty \qquad \forall x \in K .$$

Let

(14) $$Q_i(\chi_i, \xi_i) = \text{Maximum} \quad \pi_i(\xi_i, \chi_i)(\xi_i - \chi_i)$$

$$\text{subject to} \quad -q_i^- \leq \pi_i(\xi_i, \chi_i) \leq q_i^+$$

(15) <u>Proposition</u>: $$Q(\chi, \xi) = \sum_{i=1}^{\bar{m}} Q_i(\chi_i, \xi_i) .$$

The optimal solution to (14), and so to (12) can be obtained as follows:

If $(\xi_i - \chi_i) < 0$, set $\pi_i(\xi_i, \chi_i) = -q_i^-$ i.e. the coefficient of the objective function is negative, we set $\pi_i(\xi_i, \chi_i)$ at its lowest possible value because we are maximizing.

If $(\xi_i - \chi_i) > 0$, set $\pi_i(\xi_i, \chi_i) = q_i^+$.

If $(\xi_i - \chi_i) = 0$, take for $\pi_i(\xi_i, \chi_i)$ any value of the interval $[-q_i^-, q_i^+]$.

Let

$$\pi_i(\chi_i) = E_\xi \{\text{optimal} \quad \pi_i(\xi_i, \chi_i)\}$$

be the expected value of the optimal solution to (14). If ξ_i has a continuous density function, then $\pi_i(\chi_i)$ is unique, but not if $\text{Prob}\{\xi_i = \chi_i\} > 0$. By definition we set $\pi_i(\chi_i) = -q_i^-$ when $(\xi_i - \chi_i) = 0$, but we come back to this problem in the last section (IV).

In what follows we assume that $q^+ + q^- = \tilde{q} \geq 0$ otherwise our problem would be without interest. If we assume that the second stage problem is

bounded, then the optimal solution to (11) must satisfy the condition $y^+y^- = 0$ (i.e. $y_i^+ > 0 \to y_i^- = 0$ and $y_i^- > 0 \to y_i^+ = 0$). One could then show that $Q(\chi)$ is convex iff $\tilde{q} \geq 0$, using e.g. the property that a function $Q_i(\chi_i)$ is convex iff it has non-decreasing first differences and that $Q(\chi)$ is a convex combination of convex functions.

Let

$$\pi(\chi) = (\pi_1(\chi_1), \ldots, \pi_i(\chi_i), \ldots, \pi_{\bar{m}}(\chi_{\bar{m}}))$$

$$Q_i(\chi) = E_{\xi_i} \{Q_i(\chi_i, \xi_i)\}$$

$$Q(\chi) = E_{\xi}\{Q(\chi, \xi)\} .$$

Since the expectation of a sum of random variables equals the sum of the expectation of these random variables and using (15) we have

(16) <u>Proposition</u>: $Q(\chi) = \sum_{i=1}^{\bar{m}} Q_i(\chi_i)$

Since the different $Q_i(\chi_i)$ are convex, we have now proved the separability of $Q(\chi)$. From the duality theory for linear programming, we also get

$$P(\chi, \xi) = Q(\chi, \xi) \qquad \forall \text{ given pair } \chi \text{ and } \xi ,$$

then

$$P(\chi) = E_{\xi}\{P(\chi, \xi)\} = E_{\xi}\{Q(\chi, \xi)\} = Q(\chi) .$$

B. <u>A Study of</u> $Q_i(\chi_i)$.

We point out some of the characteristics of the functions $Q_i(\chi_i)$, which are useful to simplify the computation procedures when seeking an optimal

solution and also to obtain explicit forms for the equivalent convex programming problem when the ξ_1 's have some specific distribution functions.

By definition

$$\pi_1(x_1) = -q_1^- \int_{\xi_1 \leq x_1} dF_1(\xi_1) + q_1^+ \int_{\xi_1 > x_1} dF_1(\xi_1)$$

(17)

$$= q_1^+ - \tilde{q}_1 \int_{\xi_1 \leq x_1} dF(\xi_1)$$

where

$$\tilde{q}_1 = q_1^+ + q_1^-$$

$F_1(\xi_1)$ is the distribution function of ξ_1 .

Also

$$Q_1(x_1) = -q_1^- \int_{\xi_1 \leq x_1} (\xi_1 - x_1) dF_1(\xi_1) + q_1^+ \int_{\xi_1 > x_1} (\xi_1 - x_1) dF_1(\xi_1)$$

$$= q_1^+ \int_{\xi_1 \in \Xi_1} (\xi_1 - x_1) dF_1(\xi_1) - \tilde{q}_1 \int_{\xi_1 \leq x_1} (\xi_1 - x_1) dF_1(\xi_1)$$

$$= q_1^+ \bar{\xi}_1 - \tilde{q}_1 \int_{\xi_1 \leq x_1} \xi_1 dF_1(\xi_1) - \left[q_1^+ x_1 - \tilde{q}_1 \int_{\xi_1 \leq x_1} x_1 dF_1(\xi_1) \right] .$$

We write

(18) $$Q_1(x_1) = q_1^+ \bar{\xi}_1 - \psi_1(x_1) - \pi_1(x_1) x_1$$

where

$$\bar{\xi}_i = E_{\xi_i}\{\xi_i\}$$

$$(19) \qquad \psi_i(\chi_i) = \tilde{q}_i \int_{\xi_i \leq \chi_i} \xi_i \, dF_i(\xi_i)$$

then

$$Q(\chi) = \sum_{i=1}^{\bar{m}} Q_i(\chi_i) = \sum_{i=1}^{\bar{m}} q_i^+ \bar{\xi}_i - \sum_{i=1}^{\bar{m}} \left[\psi_i(\chi_i) + \pi_i(\chi_i)\, \chi_i \right].$$

In order to obtain a more explicit form of $Q_i(\chi_i)$ we divide the range of χ_i in three parts, $(-\infty, \alpha_i)$, $[\alpha_i, \beta_i]$, $(\beta_i, +\infty)$ and we express $Q_i(\chi_i)$ for these intervals. If ξ_i has no lower bound, we set $\alpha_i = -\infty$ and consider the first interval empty, if ξ_i has no upper bound we set $\beta_i = +\infty$ and the third interval is then empty.

Case 1. $\chi_i < \alpha_i$ then $\{\xi_i \mid \xi_i \leq \chi_i\} = \emptyset$.

In this region:

$$\pi_i(\chi_i) = q_i^+$$

$$\psi_i(\chi_i) = 0$$

$$Q_i(\chi_i) = q_i^+ \bar{\xi}_i - q_i^+ \chi_i$$

and

$$\frac{d}{d\chi_i} Q_i(\chi_i) = -q_i^+ \qquad \text{on } (-\infty, \alpha_i)$$

$$= -\pi_i(\chi_i) \qquad \text{on } (-\infty, \alpha_i).$$

Thus, the function $Q_1(\chi_1)$ is linear on the interval $(-\infty, \alpha_1)$. As mentioned above, this interval may be empty. (See Appendix I).

Case 2. $\alpha_1 \leq \chi_1 \leq \beta_1$ then $\{\xi_1 \mid \xi_1 \leq \chi_1\} = \{\xi_1 \mid \alpha_1 \leq \xi_1 \leq \chi_1\}$.

In this region

$$\pi_1(\chi_1) = q_1^+ - \tilde{q}_1 \int_{\alpha_1}^{\chi_1} dF_1(\xi_1)$$

$$\psi_1(\chi_1) = \tilde{q}_1 \int_{\alpha_1}^{\chi_1} \xi_1 \, dF_1(\xi_1)$$

$$Q_1(\chi_1) = q_1^+ \bar{\xi}_1 - q_1^+ \chi_1 - \tilde{q}_1 \int_{\alpha_1}^{\chi_1} (\xi_1 - \chi_1) dF_1(\xi_1) .$$

The "form" of the function $Q_1(\chi_1)$ on this interval $[\alpha_1, \beta_1]$ depends on $dF_1(\xi_1)$. In Section III of this report, we give examples for a few common distributions. If $Q_1(\chi_1)$ is differentiable on this interval, we have:

$$\frac{d}{d\chi_1} Q_1(\chi_1) = -q_1^+ + \tilde{q}_1 \int_{\alpha_1}^{\chi_1} dF_1(\xi_1) \qquad \text{on } (\alpha_1, \beta_1)$$

$$= -\pi_1(\chi_1) \qquad \text{on } (\alpha_1, \beta_1) .$$

Case 3. $\chi_1 > \beta_1$ then $\{\xi_1 \mid \xi_1 \leq \chi_1\} = \Xi_1$

In this region

$$\pi_1(\chi_1) = q_1^+ - \tilde{q}_1 = -q_1^-$$

$$\psi_1(\chi_1) = \tilde{q}_1 \bar{\xi}_1$$

$$Q_1(\chi_1) = q_1^+ \bar{\xi}_1 - \tilde{q}_1 \bar{\xi}_1 + q_1^- \chi_1 = -q_1^- \bar{\xi}_1 + q_1^- \chi_1$$

and

$$\frac{d}{dX_1} Q_1(X_1) = q_1^- \qquad \text{on } (\beta_1, +\infty)$$

$$= -\pi_1(X_1) \qquad \text{on } (\beta_1, +\infty).$$

The function $Q_1(X_1)$ is thus linear on the interval $(\beta_1, +\infty)$.

(20) Proposition: $Q_1(X_1)$ is continuous.

If $F_1(\xi_1)$ is a continuous distribution function, it is obvious to remark that $Q_1(X_1)$ is continuous at all interior points of the intervals $(-\infty, \alpha_1]$, $[\alpha_1, \beta_1]$, $[\beta_1, +\infty)$. Since $\text{Prob}\{\xi_1 = \alpha_1\} = \text{Prob}\{\xi_1 = \beta_1\} = 0$, $Q_1(X_1)$ is also continuous at α_1 and β_1. It suffices to show that $Q_1(X_1)$ is continuous for X_1 equal to a discontinuity point of $F_1(\xi_1)$. Without loss of generality, we can assume that $\text{Prob}\{\xi_1 = \alpha_1\} = f > 0$.

When X_1 converges to α_1 from the left, we have:

$$\lim_{X_1 \to \alpha_1} Q_1(X_1) = q_1^+ \bar{\xi}_1 - q_1^+ \alpha_1 .$$

When X_1 converges to α_1 from the right, we have:

$$\lim_{X_1 \to \alpha_1} Q_1(X_1) = \lim_{X_1 \to \alpha_1} q_1^+ \bar{\xi}_1 - q_1^+ X_1 - \tilde{q}_1 \int_{\alpha_1}^{X_1} (\xi_1 - X_1) dF_1(\xi_1)$$

$$= q_1^+ \bar{\xi}_1 - q_1^+ \alpha_1 - \lim_{X_1 \to \alpha_1} \tilde{q}_1 \int_{\alpha_1}^{X_1} (\xi_1 - X_1)\, dF_1(\xi_1)$$

$$= q_1^+ \bar{\xi}_1 - q_1^+ \alpha_1$$

Since the two limits are equal, $Q_i(X_i)$ is continuous at α_i.

The following figure gives the general form of $Q_i(X_i)$ where we assumed that $F_i(\xi_i)$ had discontinuities for ξ_i equal to α_i, β_i, k.

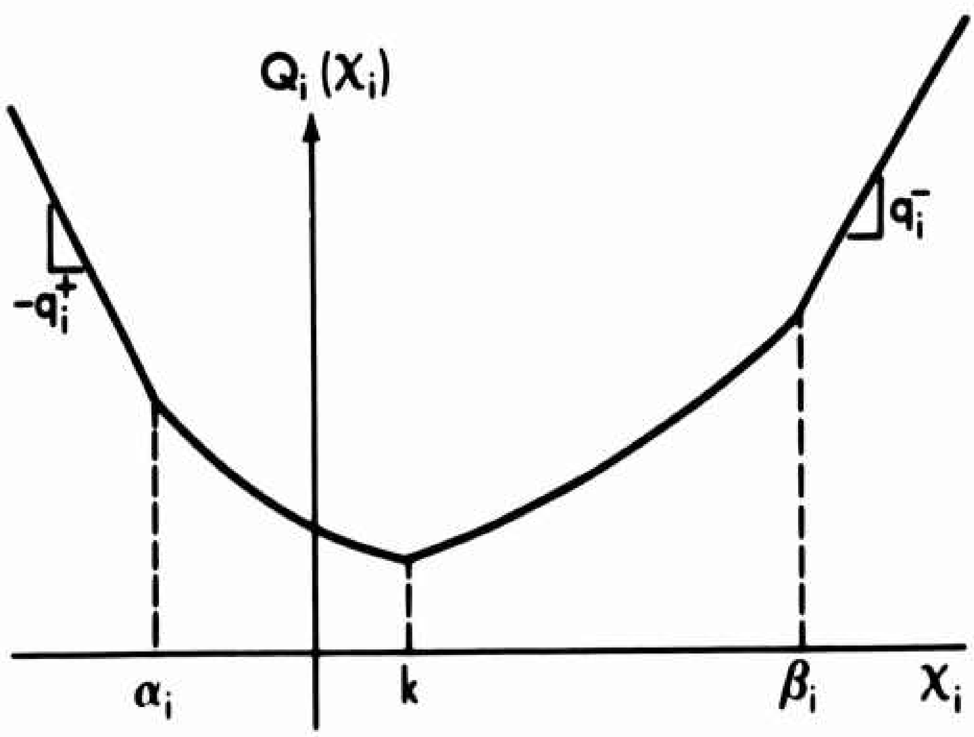


Figure I

(21) <u>Proposition</u>: If $F_i(\xi_i)$ is a continuous distribution function, then $Q_i(X_i)$ is differentiable and

$$\frac{d}{dX_i} Q_i(X_i) = -\pi_i(X_i) \qquad \text{on } R .$$

Since $F_i(\xi_i)$ is continuous, then the derivative is well determined at all interior points of $(-\infty, \alpha_i]$, $[\alpha_i, \beta_i]$, $[\beta_i, +\infty)$. Moreover, $Q_i(\chi_i)$ is continuous and at α_i and β_i, the left and the right hand derivatives are equal. This determines $\frac{d}{\chi_i} Q_i(\chi_i)$ at α_i and β_i uniquely.

The figure below indicates the general form of $Q_i(\chi_i)$ when $F_i(\xi_i)$ is a continuous distribution function.

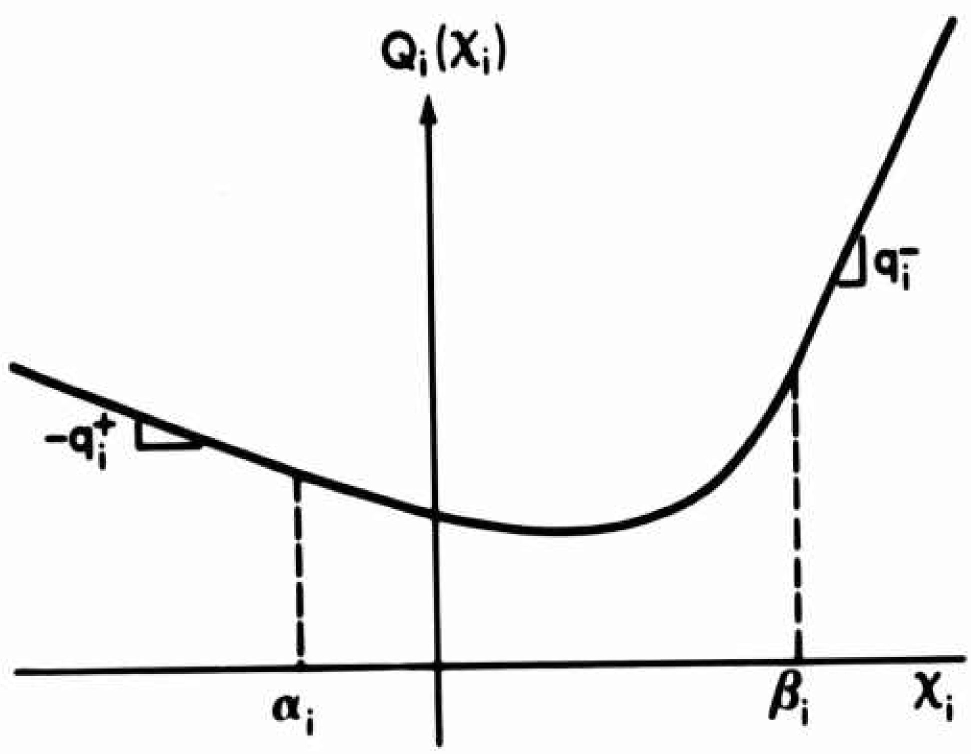


Figure II

In what follows, we assume that $\alpha_i > -\infty$. (In the Appendix I, we give the necessary modifications when α_i does not exist.)

Let

$$x_i = \bar{\xi}_i - x_{i1} + x_{i2} + x_{i3}$$

with

$$0 \leq \bar{\xi}_i - \alpha_i \leq x_{i1}$$

$$0 \leq x_{i2} \leq \beta_i - \alpha_i$$

$$0 \leq x_{i3}$$

This yields:

$$T_i x + x_{i1} - x_{i2} - x_{i3} = \bar{\xi}_i \qquad i = 1, \ldots\ldots, \bar{m} .$$

We set

$$Q_i(x_{i1}, x_{i2}, x_{i3}) = Q_i(x_i)$$

then

$$Q_i(x_{i1}, x_{i2}, x_{i3}) = q_i^+ x_{i1} + q_i^- x_{i3} + \phi_i(x_{i2}) \tag{22}$$

subject to $\bar{\xi}_i - \alpha_i \leq x_{i1}$

$$0 \leq x_{i2} \leq \beta_i - \alpha_i$$

$$0 \leq x_{i3}$$

where

$$\phi_i(x_{i2}) = -q_i^+ x_{i2} + \tilde{q}_i \int_0^{x_{i2}} (x_{i2} - \xi_i) d\hat{F}_i(\xi_i) \tag{23}$$

and

$$\hat{F}_i(\xi_i) = F_i(\xi_i + \alpha_i) .$$

Since the first term $(-q_i^+ \chi_{i2})$ of $\phi_i(\chi_{i2})$ is linear, $\tilde{q}_i \geq 0$ and $\int_0^{\chi_{i2}} (\chi_{i2} - \xi_i) d\hat{F}(\xi_i)$ is convex, so is $\phi_i(\chi_{i2})$ (over its domain). The two first terms of $Q_i(\chi_{i1}, \chi_{i2}, \chi_{i3})$ represents the linear sections of $Q_i(\chi_i)$, see Figure I. The term $Q_i(\chi_{i2})$ gives to the function its particular character, which depends on $dF_i(\xi_i)$. As we shall see in III, $\phi_i(\chi_{i2})$ may be a piece-wise linear function, a quadratic function, and so on. Let us also remark that the function $Q_i(\chi_{i1}, \chi_{i2}, \chi_{i3})$ is convex-separable, the equivalent convex programming problem to (3), in terms of $x_j, \chi_{i1}, \chi_{i2}, \chi_{i3}$ is thus a separable convex programming problem, linear in $x_j, \chi_{i1}, \chi_{i3}$. It reads:

(24) Minimize $$z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{\bar{m}} [q_i^+ \chi_{i1} + q_i^- \chi_{i3}] + \sum_{i=1}^{\bar{m}} \phi_i(\chi_{i2})$$

subject to $$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1,\ldots,m$$

$$\sum_{j=1}^{n} t_{ij} x_j + \chi_{i1} - \chi_{i3} - \chi_{i2} = \bar{\xi}_i \qquad i = 1,\ldots,\bar{m}$$

$$0 \leq x_j, \quad \bar{\xi}_i - \alpha_i \leq \chi_{i1}, \quad 0 \leq \chi_{i3}, \quad 0 \leq \chi_{i2} \leq \beta_i - \alpha_i .$$

C. Separable Convex Programming Algorithms

Two basic references in this area are [2, pp. 482-490] and [5, pp. 89-100]. In his book [2], Dantzig suggests two approaches to these problems: the bounded-variable method (or broken line fit) and the variable-coefficient method. A broken line fit to the $\phi_i(\chi_{i2})$'s would reduce our problem (24) to a large linear program (the number of variables with bounds would increase). This is

equivalent to the assumption that the distribution of ξ_i can be approximated by, or is, a discrete distribution, ξ_i taking on positive probability at the points where there is a change in the slope of the broken line fit. See III, A.

If one uses the variable coefficient approach one should take advantage of the fact that (24) is a linear programming problem, but for χ_{i2}, $i = 1,\ldots,\bar{m}$. The problem then becomes:

$$\text{(25)} \qquad \text{Minimize} \quad z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{\bar{m}} [q_i^+ \chi_{i1} + q_i^- \chi_{i3}] + \sum_{i=1}^{\bar{m}} \lambda_i g_i$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1,\ldots,m$$

$$\sum_{j=1}^{n} t_{ij} x_j + \chi_{i1} - \chi_{i3} - \chi_{i2} = \bar{\xi}_i \quad i = 1,\ldots,\bar{m}$$

$$\chi_{i2} - \lambda_i f_i = 0 \quad i = 1,\ldots,\bar{m}$$

$$\lambda_i = 1 \quad i = 1,\ldots,\bar{m}$$

$$0 \leq x_j, \ \bar{\xi}_i - \alpha_i \leq \chi_{i2}, \ 0 \leq \chi_{i3}, \ 0 \leq \chi_{i2} \leq \beta_i - \alpha_i$$

and

$$g_i \geq \phi_i(f_i) = -q_i^+ f_i + \tilde{q}_i \int_0^{f_i} (f_i - \xi_i) \, d\hat{F}_i(\xi_i) \quad i = 1,\ldots,\bar{m}$$

The solution method to this class of problems as well as the convergence properties are fully discussed in [2, pp. 486-490, pp. 433-438].

III. THE PROBABILITY SPACE: $(\Xi, \mathfrak{J}, F)$

In this section we derive the equivalent convex programming problem to (3), for some specific distribution functions F. Up to now, the assumption made on the distribution of ξ_i were limited to: $E\{\xi_i\}$ exists and one can compute the value of $\int_0^{\chi_{i2}} (\chi_{i2} - \xi_i) d\hat{F}_i(\xi_i)$, $\forall \chi_{i2} \in [0, \beta_i - \alpha_i]$ if $\alpha_i > -\infty$, (more generally one can integrate $\int_{\alpha_i}^{\chi_i} (\xi_i - \chi_i) dF_i(\xi_i)$, $\forall \chi_i \in [\alpha_i, \beta_i]$ i.e. the formulas of the Rieman-Stieltjes integration by parts apply). We did not require the independence of the ξ_i.

A. Ξ is finite.

The notations used in this paragraph differ slightly from the previous section.

Let $\xi_i^1 < \xi_i^2 \ldots\ldots < \xi_i^{k_i}$ be the values assumed by ξ_i with probabilities $f_i^1, f_i^2, \ldots, f_i^{k_i}$ respectively.

Let

$$F_i^s = \sum_{\ell=1}^{s-1} f_i^\ell = \text{Prob}\{\xi_i < \xi_i^s\}$$

$$F_i^{k_i+1} = 1 = \text{Prob}\{\xi_i < \infty\}, \quad F_i^1 = \text{Prob}\{\xi_i < \alpha_i\} = 0$$

$$\bar{\xi}_i = \sum_{\ell=1}^{k_i} \xi_i^\ell f_i^\ell = E\{\xi_i\}$$

It is easy to see that

$$E_{\xi_i}\{\text{Min} \quad q_i^+ y_i^+ + q_i^- y_i^- \mid \xi_i^s \le \chi_i \le \xi_i^{s+1}\}$$

$$= q_i^+ \sum_{\ell=s+1}^{k_i} (\xi_i^\ell - \chi_i) f_i^\ell + q_i^- \sum_{\ell=1}^{s} (\chi_i - \xi_i^\ell) f_i^\ell \ .$$

Then

$$Q_i(\chi_i) = q_i^+ \bar{\xi}_i - \sum_{\ell=1}^{k_i+1} [q_i^+ - F_i^\ell \tilde{q}_i] \chi_i^\ell$$

where $\sum_{\ell=1}^{k_i} \chi_i^\ell = \chi_i$

$$\chi_i^1 \le \xi_i^1 \qquad\qquad = d_i^1$$

$$0 \le \chi_i^\ell \le \xi_i^\ell - \xi_i^{\ell-1} = d_i^\ell \qquad\qquad = 2,\ldots,k_i$$

$$0 \le \chi_i^{k_i+1} \ .$$

Since $\tilde{q}_i \ge 0$ (the second stage problem is bounded by assumption) and

$$F_i^\ell < F_i^{\ell+1} \qquad \text{for } \ell=1,\ldots,k_i$$

it is readily seen that $Q_i(\chi_i)$ is a <u>piece-wise linear convex function.</u> This last property allows us to formulate our original problem as a linear program [2, pp. 484-485], viz.:

$$(26) \qquad \text{Minimize } z = \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{\bar{m}} \sum_{\ell=1}^{k_i+1} (q_i^+ - F_i^\ell \tilde{q}_i) \chi_i^\ell + \sum_{i=1}^{\bar{m}} q_i^+ \bar{\xi}_i$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i=1,\ldots,m$$

$$\sum_{j=1}^{n} t_{ij} x_j - \sum_{\ell=1}^{k_i} \chi_i^{\ell} = 0$$

$$x_j \geq 0 \;, \qquad j=1,\ldots,n$$

$$\chi_i^1 \leq d_i^1 \,,\; 0 \leq \chi_i^{\ell} \leq d_i^{\ell} \,,\; 0 \leq \chi_i^{k_i+1}$$

$$\text{for } i=1,\ldots,m \text{ and } \ell=2,\ldots,k_i+1$$

where $\sum_{i=1}^{\bar{m}} q_i^{+} \bar{\xi}_i$ is a constant.

$-\chi_{i1}$ in (24) corresponds to χ_i^1 and χ_{i3} in (24) corresponds to $\chi_i^{k_i+1}$. The variables χ_i^{ℓ}, $\ell=2,\ldots,k_i$ in (26) correspond to the unique variable χ_{i2} in (24).

This problem can now be solved using a linear programming code with upper-bound variable option.

1. Allocation of Aircraft to Routes under Uncertain Demand

The approach indicated above could be attributed to Ferguson and Dantzig where it was underlying their work: "Allocation of Aircraft to Routes under Uncertain Demand." [2, pp. 568-591]. Using their notation, the problem written in standard form (3) is:

$$\text{Minimize} \quad E_{\xi} \left\{ \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (c_{ij} - p_{ij} k_j) x_{ij} + \sum_{j=1}^{n-1} c_{mj} x_{mj} + \sum_{j=1}^{n-1} k_j y_j \right\}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i \qquad i=1,\dots,m-1$$

$$\sum_{i=1}^{m-1} p_{ij}x_{ij} + x_{mj} - y_j = \xi_j \qquad j=1,\dots,n-1$$

$$x_{ij} \geq 0, \quad x_{mj} \geq 0, \quad y_j \geq 0 \qquad i=1,\dots,m-1\,;\ j=1,\dots,n\,,$$

where y_j is the number of seats remaining available and ξ_j here is their d_j. The interpretation of the other symbols is given in [2, pp. 574]. This problem has the following features:

$c_{mj} = 0$ for all j

In our formulation this means $q^+ = 0$

i.e. $q_i^+ - F_i^\ell \tilde{q}_i = -F_i^\ell q_i^-$.

In their terms $-F_i^\ell\, q_i^- = -k_j(F_j^\ell) = -k_j(1 - \gamma_{hj}^*)$

$p_{ij} \geq 0$ implies $\sum_{i=1}^{m-1} p_{ij}x_{ij} \geq 0$. i.e. $\chi_j^1 \geq 0$ for all j. The random variable ξ_j on values $\xi_j^1 < \xi_j^2 \,\dots\dots < \xi_j^{k_j}$ and it is assumed that:

$$\sum_{j=1}^{m-1} p_{ij}x_{ij} \leq \xi_j^{k_j} \quad \text{for all } j$$

i.e. $\chi_j^{k_j+1}$ is fixed at value zero. Taking these modifications into account, the linear program, corresponding to our general form (26) follows:

$$\text{(27)} \qquad \text{Min} \quad \sum_{i=1}^{m-1} \sum_{i=1}^{n-1} \tilde{c}_{ij}x_{ij} + \sum_{j=1}^{n-1} k_j \sum_{\ell=1}^{k_j} F_j^\ell\, \chi_j^\ell + R_0$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{ij} = a_i \qquad i=1,\ldots,m-1$$

$$\sum_{i=1}^{m-1} p_{ij}x_{ij} - \sum_{\ell=1}^{k_j} x_j^\ell = 0$$

$$x_{ij} \geq 0 \qquad i=1,\ldots,m-1\,, \quad j = 1,\ldots,n$$

$$0 \leq x_j^\ell \leq d_j^\ell \qquad j=1,\ldots,n \quad \text{and} \quad \ell=1,\ldots,k_j$$

where

$\tilde{c}_{ij} = c_{ij} - p_{ij}k_j$ i.e. cost (negative profit) of flying aircraft type i on route j at full capacity.

$$R_0 = \sum_{j=1}^{n-1} \bar{\xi}_j k_j$$

and

d_j^ℓ is defined as above.

There the first $n \times m$ columns of the constraints matrix (matrices A and T in our standard form (3) have the structure of a weighted distribution problem, Ferguson and Dantzig specialized the upper-bound algorithm for linear programs to this class of problems which lead to an elegant solution technique, taking full advantage of the nature of the problem. We would like to point out a slight conceptual difference between Ferguson and Dantzig's formulation [2, p. 577] and ours, reflected in the objective functions. The Ferguson-Dantzig objective form can be interpreted as follows: only the costs of

flying airplane type i on route j are certain (c_{ij}) and one __expects__ a certain revenue obtained when filling up the seat capacities made available; where our objective reads as follows: profit $(\tilde{c}_{ij})$ of flying aircraft type i on route j are certain and one __expects__ only a lost revenue resulting from not filling the seat-capacity made available. Obviously, both objectives yield the same values for the optimal x_{ij}'s and we can derive one from the other.

2. Elmaghraby's Approach

The problem studied by Elmaghraby in "An Approach to Linear Programming under Uncertainty," [4], written in standard form, is as follows:

$$\underset{(x,y^+,y^-)}{\text{Minimize}}\ z = E_\xi\{cx + q^+y^+ + q^-y^-\}$$

$$\begin{array}{lll} \text{subject to} & Ax & \leq b \\ & Ix + Iy^+ - Iy^- & = \xi \\ & x \geq 0\ y^+ \geq 0\ y^- \geq 0 & \end{array}$$

then $\chi_i = x_i$ $(i=1,\ldots,\bar{m} = n)$ and one can speak of the objective function

$$z = cx + Q(\chi) = cx + Q(x) = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} Q_j(x_j)$$

as a separable convex function in x, rather than x and χ as before (24), but this does not lead to noticeable computational simplifications.

In what follows we present Elmaghraby's version of the linear program used to solve his problem which will obtain its solution by a "sequence" of linear programs (by this he means that $Q_j(x_j)$ can be broken up in linear

sections and the simplex method will examine these different linear sections in "sequence").

$$\underset{x_j^\ell}{\text{Minimize}} \quad z = \sum_{j=1}^{n} \sum_{\ell=1}^{k_j+1} [c_j - q_j^+ - F_j^\ell \tilde{q}_j] x_j^\ell$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} \left(\sum_{\ell=1}^{k_j+1} x_j^\ell \right) = b_i \qquad i=1,\ldots,m$$

$$\sum_{\ell=1}^{s_j} x_j^\ell \leq \xi_j^{s_j} \qquad \text{for } j=1,\ldots,n \; ; \; s_j=1,\ldots,k_j$$

$$x_j^\ell \geq 0 \qquad j=1,\ldots,n \; ; \; \ell=1,\ldots,k_j+1$$

$$\text{where} \quad x_j = \sum_{\ell=1}^{k_j+1} x_j^\ell .$$

If $\overset{o}{x}{}_j^\ell$ is the optimal solution to the problem then $\overset{o}{x}_j = \sum_{j=1}^{k_j} \overset{o}{x}{}_j^\ell$ is optimal for his original problem. It is obvious that the inequalities $\sum_{\ell=1}^{s_j} x_j^\ell \leq \xi_j^{s_j}$ could have been used to obtain upper-bounds for x_j^ℓ as was done for x_1^ℓ in (26). This reduces the size of the problem considerably.

3. El-Agizy's Approach

An alternate method to reduce problem (3) to the linear programming problem (26) is given in El-Agizy [3]. This derivation gives also an

alternative proof that the assumption of independence of the ξ_i 's is superfluous.

B. $\underline{\xi_i \text{ is uniform, } \forall\, i}$

Let

$$f_i(\xi_i) = \frac{1}{\beta_i - \alpha_i} \qquad \text{if } \xi_i \in [\alpha_i, \beta_i]$$

$$= 0 \qquad \text{otherwise}$$

then

$$\hat{f}_i(\xi_i) = \frac{1}{\beta_i - \alpha_i} \qquad \text{if } \xi_i \in [0, \beta_i - \alpha_i]$$

$$= 0 \qquad \text{otherwise}$$

and by (23)

$$\phi_i(x_{i2}) = -q_i^+ x_{i2} + \frac{\tilde{q}_i}{\beta_i - \alpha_i} \int_0^{x_{i2}} (x_{i2} - \xi_i)\, d\xi_i = -q_i^+ x_{i2} + \frac{\tilde{q}_i}{2(\beta_i - \alpha_i)} x_{i2}^2$$

(24) becomes:

$$\text{(28)} \quad \text{Minimize } z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{\bar{m}} (q_i^+ x_{i1} - q_i^+ x_{i2} + q_i^- x_{i3}) + \frac{1}{2} \sum_{i=1}^{\bar{m}} \frac{\tilde{q}_i}{\beta_i - \alpha_i} x_{i2}^2$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i=1,\ldots,m$$

$$\sum_{j=1}^{n} b_{ij} x_j + x_{i1} - x_{i2} - x_{i3} = \bar{\xi}_i \qquad i=1,\ldots,\bar{m}$$

$$0 \le x_j, \quad \bar{\xi}_i - \alpha_i \le x_{i1}, \quad 0 \le x_{i2} \le \beta_i - \alpha_i, \quad 0 \le x_{i3}$$

(23) is easily recognizable as a Quadratic Programming Problem for which many algorithms exist in the literature, e.g. see [2, pp. 490-497]. Beale was the first one to point out this property for uniform distribution [1].

C. ξ_i is exponential, $\forall i$

Let

$$f_i(\xi_i) = \lambda_i e^{-\lambda_i \xi_i} \qquad \text{if } \xi_i \in [0, +\infty]$$

$$= 0 \qquad \text{otherwise}$$

then

$$\hat{f}_i(\xi_i) = f_i(\xi_i), \qquad \bar{\xi}_i = \frac{1}{\lambda_i}$$

and by (23)

$$\phi_i(x_{i2}) = -q_i^+ x_{i2} + \tilde{q}_i \lambda_i \int_0^{x_{i2}} (x_{i2} - \xi_i) e^{-\lambda_i \xi_i} d\xi_i$$

$$= q_i^- x_{i2} - \frac{\tilde{q}_i}{\lambda_i} (1 - e^{-\lambda_i x_{i2}})$$

using Taylor's expansion

$$= q_i^- x_{i2} - \frac{\tilde{q}_i}{\lambda_i} \left(1 - 1 + x_{i2}\lambda_i - \frac{x_{i2}^2 \lambda_i^2}{2} + \sum_{n=3}^{\infty} (-1)^n \frac{x_{i2}^n \lambda_i^n}{n!}\right)$$

$$= -q_i^+ x_{i2} + \frac{\tilde{q}_i \lambda_i}{2} x_{i2}^2 + \tilde{q}_i \sum_{n=3}^{\infty} (-\lambda_i)^{n-1} \frac{x_{i2}^n}{n!}$$

an approximation to $\phi_i(x_{i2})$

$$\simeq -q_i^+ x_{i2} + \frac{\tilde{q}_i \lambda_i}{2} x_{i2}^2 .$$

The value of this approximation depends on the relative value of $\tilde{q}_i$ and the proximity of the optimal value of χ_{i2} to $\bar{\xi}_i = \frac{1}{\lambda_i}$. If we introduce the approximation of $\phi_i(\chi_{i2})$ in the objective function of (24), the resulting equivalent convex programming problem is:

$$\text{(29)} \qquad \text{Minimize } z = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{\bar{m}} \left[q_i^+ \chi_{i1} - q_i^+ \chi_{i2} \right] + \frac{1}{2} \sum_{i=1}^{\bar{m}} \tilde{q}_i \lambda_i \chi_{i2}^2$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i=1,\ldots,m$$

$$\sum_{j=1}^{n} t_{ij} x_j + \chi_{i1} - \chi_{i2} = \frac{1}{\lambda_i} \qquad i=1,\ldots,\bar{m}$$

$$0 \leq x_j, \quad \frac{1}{\lambda_i} \leq \chi_{i1}, \quad 0 \leq \chi_{i2}.$$

So as (28) this is a Quadratic Programming Problem ($\forall$ i, $\tilde{q}_i \lambda_i \geq 0$). Remark that we did not introduce χ_{i3}, because $\beta_i = +\infty$, i.e. ξ_i has no upper bound.

D. ξ_i has a continuous distribution function, $\forall$ i

In our last paragraph (III.C) we "accepted" an approximation to the objective function in order to reduce (24) - the equivalent convex programming to (3) - to a quadratic programming problem for which algorithms have been developed. The purpose of this paragraph is to suggest approximation for the distribution functions and then show that the so obtained "equivalent" convex programming problem is in a form for which efficient computational

methods exist.

We have already pointed out in Section II.C (on Separable Convex Programming) that replacing $\phi_i(x_{i2})$ by a broken line fit is equivalent to finding a discrete distribution which would "approximate", in some sense, the distribution of the random variable ξ_i. Here, we approximate continuous distribution by step-functions. In other words, we replace the random variable ξ_i by a weighted sum of random variables having uniform distributions.

Set

$$\xi_i = \sum_{\ell=1}^{k_i} p_i^\ell \xi_i^\ell$$

where

$$\sum_{\ell=1}^{k_i} p_i^\ell = 1$$

$f_i^\ell(\xi_i^\ell)$ $\ell=1,\ldots,k_i$ are uniform density functions.

In (24), replace the constraint

$$\sum_{j=1}^{n} t_{ij}x_j + x_{i1} - x_{i2} - x_{i3} = \bar{\xi}_i$$

by k_i equations of the form

$$\sum_{j=1}^{n} t_{ij}x_j + x_{i1}^\ell - x_{i2}^\ell - x_{i3}^\ell = \bar{\xi}_i^\ell \qquad \ell=1,\ldots,k_i.$$

The objective function of (24) becomes

$$z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{\bar{m}} \sum_{\ell=1}^{k_i} p_i^{\ell}(q_i^+ \chi_{i1}^{\ell} + q_i^- \chi_{i3}^{\ell}) + \sum_{i=1}^{\bar{m}} \sum_{\ell=1}^{k_i} p_i^{\ell} \phi_i^{\ell}(\chi_{i2}^{\ell}) .$$

We have already shown (III. C) that if ξ_i^{ℓ} is uniform then $\phi_i^{\ell}(\chi_{i2}^{\ell})$ has a linear and a quadratic term. See (28). Then

$$z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{\bar{m}} \sum_{\ell=1}^{k_i} p_i^{\ell} [q_i^+ \chi_{i1}^{\ell} - q_i^+ \chi_{i2}^{\ell} + q_i^- \chi_{i3}^{\ell}] + \frac{1}{2} \sum_{i=1}^{\bar{m}} \sum_{\ell=1}^{k_i} p_i^{\ell} \frac{\tilde{q}_i}{\beta^{\ell} - \alpha_i^{\ell}} (\chi_{i2}^{\ell})^2 .$$

This approximation of random variables having continuous distribution by the sum of random variables having uniform distributions led also to a Quadratic Programming Problem. It is clear that the increase in size $(3\,\bar{m} \sum_{i=1}^{\bar{m}} (k_i - 1)$ new variables of which $2\bar{m} \sum_{i=1}^{\bar{m}} (k_i - 1)$ are bounded and $(k_i - 1)\bar{m}$ additional constraints) depends on the desired quality of the approximation. To find the α_i^{ℓ} and β_i^{ℓ}, lower and upper bounds for ξ_i^{ℓ}, see the Appendix to [1].

E. Summary

This section has shown that either directly or by approximation it was sometimes possible to reduce the equivalent convex programming to (3), to programming problems for which we possess efficient algorithms. For simplicity we have assumed in each paragraph that the marginal density function $f_i(\xi_i)$ was of the same nature $\forall\, i$. This is not necessarily the case. It should be clear by now that each $\phi_i(\chi_{i2})$ can be treated independently. For instance, if ξ_1 has a discrete distribution, and say ξ_2 a uniform distribution it is not difficult to show that the equivalent convex programming problem is a quadratic programming problem.

IV. AN ALGORITHM FOR CONTINUOUS DISTRIBUTION FUNCTIONS

We now give an algorithm to solve problem (3) when $\forall_i$, $F_i(\xi_i)$ is a continuous distribution function. We assume that the distribution functions $F_i(\xi_i)$ allow Rieman-Stieltjes integration of linear functions of ξ_i. We also assume that (3) is solvable which implies among other conditions that $\ddot{q} \geq 0$. We have shown (4) that the equivalent convex programming to (3) can be written

$$(30)\qquad \begin{aligned} \text{Minimize}\quad z(x) &= cx + Q(x) \\ \text{subject to}\quad & Ax = b \\ & x \geq 0 \end{aligned}$$

or

$$(30')\qquad \begin{aligned} \text{Minimize}\quad z(x,\chi) &= cx + Q(\chi) \\ & Ax = b \\ & Tx - \chi = 0 \\ & x \geq 0 \end{aligned}$$

where

$$Q(\chi) = \sum_{i=1}^{\bar{m}} Q_i(\chi_i) = \sum_{i=1}^{\bar{m}} \left[q_i^+ \bar{\xi}_i - \psi_i(\chi_i) - \pi_i(\chi_i)\chi_i \right]$$

then

$$Q(x) = \sum_{i=1}^{\bar{m}} q_i^+ \bar{\xi}_i - \sum_{i=1}^{\bar{m}} \left[\psi_i(T_i x) + \pi_i(T_i x) T_i x \right].$$

Since $\sum_{i=1}^{\bar{m}} q_i^+ \bar{\xi}_i$ is a constant, we may delete this term from the objective function of our problems. We also write $\psi(\chi) = \sum_{i=1}^{\bar{m}} \psi_i(\chi_i)$.

Problem (30) becomes

$$(31)\qquad \text{Minimize}\quad \hat{z}(x) = \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{\bar{m}} \Big[\psi_i(T_i x) + \pi_i(T_i x) T_i x\Big]$$

$$\text{subject to}\quad \sum_{j=1}^{n} a_{ij} x_j = b \qquad i=1,\dots,m$$

$$x_j \geq 0 \qquad j=1,\dots,n$$

We should note that:

If $f_i(\xi_i)$ is continuous at $\xi_i = \chi_i$, then

$$\pi_i(\chi_i) = q_i^+ - \tilde{q}_i \int_{\xi_i < \chi_i} dF_i(\xi_i) = q_i^+ - \tilde{q}_i \int_{\xi_i \leq \chi_i} dF_i(\xi_i) \,.$$

If $f_i(\xi_i) > 0$ for $\xi_i = \chi_i$ then $\int_{\xi_i < \chi_i} dF_i(\xi_i) \neq \int_{\xi_i \leq \chi_i} dF_i(\xi_i)$

and $q_i^+ - \tilde{q}_i \int_{\xi_i \leq \chi_i} dF_i(\xi_i) \leq \pi_i(\chi_i) \leq q_i^+ - \tilde{q}_i \int_{\xi_i < \chi_i} dF_i(\xi_i)$.

In this case a complete range of values exist for the expected values of optimal solutions to (14). Identical relations hold for $\psi_i(\chi_i)$.

In what follows, we assume that $F_i(\xi_i)$ is a continuous distribution, $\forall$ $i=1,\dots,\bar{m}$. The following propositions enable us to derive an algorithm to solve problem (31), and consequently problem (3).

(32) <u>Proposition</u>: $\frac{d}{dx}\hat{z}(x) = c - \pi(\chi)\,T$ (i.e. $\frac{\partial \hat{z}(x)}{\partial x_j} = c_j - \left[\pi(\chi)\,T\right]_j$) .

The result is immediate if we remark that (21) yields $\frac{d}{d\chi}\,Q(\chi) = -\pi(\chi)$ and also that $\chi = T\,x$.

(33) <u>Proposition</u>: $\left[c - \pi(\bar{\chi})\,T\right]x - \psi(\bar{\chi})$ is a supporting hyperplane of $\hat{z}(x)$ at $x = \bar{x}$ where $\bar{\chi} = T\,\bar{x}$.

In view of (32), it suffices to show that $\hat{z}(\bar{x}) = \left[c - \pi(\bar{\chi})\,T\right]\bar{x} - \psi(\bar{\chi})$ which is obvious by the definitions of $\hat{z}(x)$.

(34) <u>Proposition</u>: If $\left[c - \pi(\bar{\chi})\,T\right](x - \bar{x}) \geq 0$, $\forall x \in K$ then $\hat{z}(x)$ has a minimum at $\bar{x}$.

Since $\hat{z}(x)$ is convex, then the following inequality holds [7] :

$$\hat{z}(x) - \hat{z}(\bar{x}) \geq \left[c - \pi(\bar{\chi})\,T\right](x - \bar{x}) .$$

Moreover, by hypothesis the second term of this inequality is non-negative for all $x \in K$. This implies

$$\hat{z}(x) \geq \hat{z}(\bar{x}) \qquad \forall x \in K .$$

(35) <u>Proposition</u>: Let $x, \bar{x} \in K$ and such that $\left[c - \pi(\chi)\,T\right]x > \left[c - \pi(\chi)\,T\right]\bar{x}$ then $\exists\, x' \in (x, \bar{x}]$ such that $\hat{z}(x') < \hat{z}(x)$.

Since $\left[c - \pi(\chi)\,T\right]x > \left[c - \pi(\chi)\,T\right]\bar{x}$, we have

$$\left[c - \pi(\chi)T\right]x > \left[c - \pi(\chi)T\right](\lambda x + (1-\lambda)\bar{x}) \quad \forall \lambda \in [0,1] .$$

If $\hat{z}(x) \leq \hat{z}(\lambda x + (1-\lambda)\bar{x}) \quad \forall \lambda \in [0,1]$, consider

$$\zeta(\lambda) = \hat{z}(\lambda x + (1-\lambda)\bar{x}) \quad \text{where} \quad \lambda \in [0,1] .$$

Since $z(x)$ is differentiable, so is $\zeta(\lambda)$ [6] . Then

$$\frac{d}{d\lambda} \zeta(\lambda)_{\lambda=1} = [c - \pi(\chi) T] (x - \bar{x}) > 0 .$$

This implies that $\exists \lambda^{*} \in [0,1)$ such that

$$\zeta(\lambda^{*}) < \zeta(1) .$$

Let

$x^{*} = \lambda^{*} x + (1-\lambda^{*})\bar{x}$ we have $\hat{z}(x^{*}) < \hat{z}(x)$ which contradicts

$$\hat{z}(x) \leq \hat{z}(\lambda x + (1-\lambda)\bar{x}) , \quad \forall \lambda \in [0,1]$$

(36) <u>Proposition</u>: Let $x \in K$ and $\hat{z}(x) > \hat{z}(x^{*}) = \underset{x \in K}{\text{Minimum}} \ \hat{z}(x)$, then

$\exists \ \bar{x}$ such that $[c - \pi(\chi) T] x > [c - \pi(\chi) T]\bar{x}$.

Since $\hat{z}(x)$ is convex and by our hypothesis we have

$$0 > \hat{z}(x^{*}) - \hat{z}(x) \geq [c - \pi(\chi) T] (x^{*} - x) .$$

The last two propositions suggest an iterative procedure, the next proposition gives us a test of optimality.

(37) <u>Proposition</u>: $\hat{z}(x^{*}) = \underset{x \in K}{\text{Minimum}} \ \hat{z}(x)$ iff

$$[c - \pi(\chi^{*})T]x^{*} = \underset{x \in K}{\text{Minimum}} \ [c - \pi(\chi^{*})T]x \quad \text{where} \quad \chi^{*} = Tx^{*}.$$

Let $[c - \pi(\chi^\circ)T]x^\circ \leq [c - \pi(\chi^\circ)T]x \quad \forall x \in K$ then by (34) x° is optimal.

Let $\hat{z}(x^\circ) \leq \hat{z}(x) \quad \forall x \in K$ and assume that $\exists x \in K$ such that

$$[c - \pi(\chi^\circ)T]x^\circ > [c - \pi(\chi^\circ)T]x$$

then by proposition (35), $\exists\, \bar{x} \in (x^\circ, x]$ such that $\hat{z}(\bar{x}) < \hat{z}(x^\circ)$, which contradicts the assumption: $\exists x \in K$ such that $[c - \pi(\chi^\circ)T]x^\circ > [c - \pi(\chi^\circ)T]x$.

Let us now consider the following linear programming problem.

$$\text{(38)} \qquad \text{Minimize} \quad [c - \pi(\chi^k)T]\bar{x}$$

$$\text{subject to} \qquad A\bar{x} = b$$

$$\bar{x} \geq 0$$

where

$$\chi^k = Tx^k \ , \quad x^k \in K \ .$$

Since problem (31) is solvable, so is problem (38) $\forall x^k \in K$; (proposition (20) and the linearity of the term cx proves the continuity of $\hat{z}(x)$ over K).

By (37), if x^k is an optimal solution to (38), then x^k is optimal for (31). If x^k is not an optimal solution, then by (36) the optimal solution to (38), say $\bar{x}^k$, is such that

$$[c - \pi(\chi^k)T](x^k - \bar{x}^k) > 0$$

then by (35), $\exists\, x^{k+1} \in (x^k, \bar{x}^k]$ such that

$$\hat{z}(x^{k+1}) < \hat{z}(x^k) \ .$$

Since $x^{k+1} \in K$, we can find $\pi(\chi^{k+1})$ and solve a new linear program of the

form (38) where we introduce the new values for the row vector $\pi(\chi)$. To find x^{k+1} consider the function:

$$(39) \qquad \zeta(\lambda) = \hat{z}(\lambda x^k + (1-\lambda)\bar{x}^k) \qquad \lambda \in [0,1]$$

$$= \lambda \sum_{j=1}^{n} c_j (x_j^k - \bar{x}_j^k) + \sum_{j=1}^{n} c_j x_j^k - \sum_{i=1}^{\bar{m}} \phi_i (\lambda(\chi_i^k - \bar{\chi}_i^k) + \bar{\chi}_i^k)$$

$$- \sum_{i=1}^{\bar{m}} \pi_i (\lambda(\chi_i^k - \bar{\chi}_i^k) + \bar{\chi}_i^k)(\lambda(\chi_i^k - \bar{\chi}_i^k) + \bar{\chi}_i^k) .$$

Since $\hat{z}(x)$ is differentiable, so is $\zeta(\lambda)$ [6]. The derivative of $\zeta(\lambda)$ with respect to λ for $\lambda_s \leq \lambda \leq \lambda_{s+1}$, is

$$\frac{d}{d\lambda}\zeta(\lambda) = c(x^k - \bar{x}^k) - \sum_{i=1}^{\bar{m}} q_i^+ (\chi_i^k - \bar{\chi}_i^h) + \sum_{i \in I_2^s} \tilde{q}_i (\chi_i^k - \bar{\chi}_i^k) \int_{\alpha_i}^{\lambda(\chi_i^k - \bar{\chi}_i^k) + \bar{\chi}_i^k} dF_i(\xi_i)$$

$$+ \sum_{i \in I_3^s} \tilde{q}_i (\chi_i^k - \bar{\chi}_i^k)$$

where

$$I_2^s = \left\{ i \,\middle|\, \frac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \leq \lambda_s < \lambda_{s+1} \leq \frac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \right\}$$

$$I_3^s = \left\{ i \,\middle|\, \frac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \leq \lambda_s \right\}$$

and

$$\{\lambda_s\} = \left\{ 0, 1, \frac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}, \frac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}, \quad i = 1, \ldots, \bar{m} \right\} \cap [0,1]$$

and

$$\lambda_s < \lambda_{s+1} \qquad \forall s = \{1,2,\ldots, r \leq 2\bar{m} + 2\}.$$

We assumed here that $\chi_i^k - \bar{\chi}_i^k > 0$, this is not the case $\forall_i$, we develop the derivation of $\frac{d}{d\lambda}\zeta(\lambda)$ in more detail in Appendix II. To find the minimum of $\zeta(\lambda)$ we successively compute the value of $\frac{d}{d\lambda}\zeta(\lambda)$ at the points λ_s (at most $2\bar{m} + 2$), for $s=1,\ldots,r$.

If $\frac{d}{d\lambda}\zeta(0) \geq 0$ then $\zeta(\lambda)$ attains its minimum on $[0,1]$ at $\lambda = 0$.

If $\frac{d}{d\lambda}\zeta(\lambda_s) \leq 0$ and $\frac{d}{d\lambda}\zeta(\lambda_{s+1}) \geq 0$ then $\zeta(\lambda)$ attains its minimum

at some $\lambda \in [\lambda_s, \lambda_{s+1}]$.

If $\frac{d}{d\lambda}\zeta(1) \leq 0$ then $\zeta(\lambda)$ attains its minimum on $[0,1]$ at $\lambda = 1$.

If $\zeta(\lambda)$ attains its minimum at $\lambda = 1$, then

$$\hat{z}(x^k) \leq \hat{z}(x) \qquad \forall x \in [x^k, \bar{x}^k].$$

This implies that x^k was an optimal solution to (38), otherwise we contradict (36), thus x^k is an optimal solution to (31). Let λ^k be the minimum of $\zeta(\lambda) = \hat{z}(\lambda x^k + (1-\lambda)\bar{x}^k)$ on $[0,1)$ we set

$$x^{k+1} = \lambda^k x^k + (1-\lambda^k)\bar{x}^k.$$

A flow chart of this algorithm is given at the end. We now show the convergence of this process. Propositions (35) and (36) assure us that if x^k is not an optimal solution for (31), then $z(x^k) > z(x^{k+1})$ since $z(x)$ attains its minimum value on $[x^k, \bar{x}^k]$ at x^{k+1}. Moreover, problem (31) being solvable implies that the series $\{z(x^k)\}$ is Cauchy convergent.

(41) <u>**Proposition:**</u> $[c - \pi(\chi^k)T]\,(\bar{x}^k - x^k) \leq \hat{z}(x^\circ) - \hat{z}(x^k)$ where x° is an optimal solution of (31).

$\hat{z}(x)$ is convex and (21) then

$$\hat{z}(x^\circ) \geq \hat{z}(x^k) \geq [c - \pi(\chi^k)T]\,(x^\circ - x^k)$$

and since $\bar{x}^k$ is optimal for (38), we have

$$[c - \pi(\chi^k)T]x^\circ \geq [c - \pi(\chi^k)T]\bar{x}^k .$$

Adding up these two inequalities gives the desired result. From this last proposition, we have obtained a lower bound for $\hat{z}(x^\circ)$ and

$$(42) \qquad \hat{z}(x^k) + [c - \pi(\chi^k)T]\,(\bar{x}^k - x^k) \leq \hat{z}(x^\circ) \leq \hat{z}(x^k) .$$

At each cycle of the algorithm, we could compute $\hat{z}(x^k) + [c - \pi(\chi^k)T](\bar{x}^k - x^k)$ and use for lower bound of $z(x^\circ)$:

$$L_k = \max_{\ell=1,\ldots,k} \left\{\hat{z}(x^\ell) + [c - \pi(\chi^\ell)T](\bar{x}^\ell - x^\ell)\right\} .$$

We obtain

$$L_k - \hat{z}(x^k) \leq \hat{z}(x^o) - \hat{z}(x^k) \leq 0$$

then $\hat{z}(x^k) - L_k$ is an upper bound on $\hat{z}(x^k) - \hat{z}(x^o)$. This upper bound could be used for stopping the computation, e.g. when $\hat{z}(x^k) - L_k$ is less than a predetermined number.

To show that $\hat{z}(x^k) \to z(x^o)$ it suffices to show that $[c - \pi(\chi^k)T](\bar{x}^k - x^k)$ has a subsequence such that $\lim_{k_i \to \infty}[c - \pi(\chi^{k_i})T](\bar{x}^{k_i} - x^{k_i}) \to 0$. If the process

is finite, we have $\bar{x}^k = x^k$, $\forall k \geq k_0$. Let us assume that $\hat{z}(x^k) > z(x^\circ)$ $\forall k$, then

(43) <u>Proposition</u>: $\nexists D > 0$ such that $[c - \pi(\chi^k)T](x^k - \bar{x}^k) > D$ $\forall k$.

Let us assume that $\exists D > 0$ such that $[c - \pi(\chi^k)T](x^k - \bar{x}^k) = \frac{d}{d\lambda}\zeta(1) > D, \forall k$ then by continuity of $\pi(\chi)$, see (20), and for $k \geq k_0$, $\exists v_k \in (0,1]$ such that

$$[c - \pi(v^k\chi^k + (1 - v^k)\bar{\chi}^k)T](x^k - \bar{x}^k) = \frac{d}{d\lambda}\zeta(v^k) = D/2.$$

Moreover, by convexity of $z(x)$, we have

$$z(x^k) - z(v^k x^k + (1 - v^k)\bar{x}^k) \geq (1 - v^k)[c - \pi(v^k\chi^k + (1 - v^k)\bar{\chi}^k)T](x^k - \bar{x}^k)$$

and we also have

$$z(v^k x^k + (1 - v^k)\bar{x}^k) - z(x^{k+1}) \geq 0.$$

Adding up these two inequalities, we obtain

$$z(x^k) - z(x^{k+1}) \geq (1 - v^k)[c - \pi(v^k\chi^k + (1 - v^k)\bar{\chi}^k)\,T](x^k - \bar{x}^k) = \frac{D}{2}(1 - v^k)$$

thus

$$(1 - v^k) \leq \frac{2}{D}[\hat{z}(x^k) - \hat{z}(x^{k+1})]$$

$$1 \geq v^k \geq \frac{2}{D}[\hat{z}(x^{k+1}) - \hat{z}(x^k)] + 1.$$

Since $\{z(x^k)\}$ is cauchy convergent, we have

$$\lim_{k \to \infty} v_k = 1.$$

We also have:

$$\frac{D}{2} < \Big([c - \pi(\chi^k)T] - [c - \pi(v^k\chi^k + (1 - v^k)\bar{\chi}^k)T] \Big)(x^k - \bar{x}^k) \leq$$

$$\| [c - \pi(\chi^k)T] - [c - \pi(v^k\chi^k + (1 - v^k)\bar{\chi}^k)T] \| \; \| x^k - \bar{x}^k \| \leq$$

$$\| [c - \pi(\chi^k)T] - [c - \pi(v^k\chi^k + (1 - v^k)\bar{\chi}^k)T] \| \cdot M$$

where $M \geq \sup_k \| x^k - \bar{x}^k \|$.

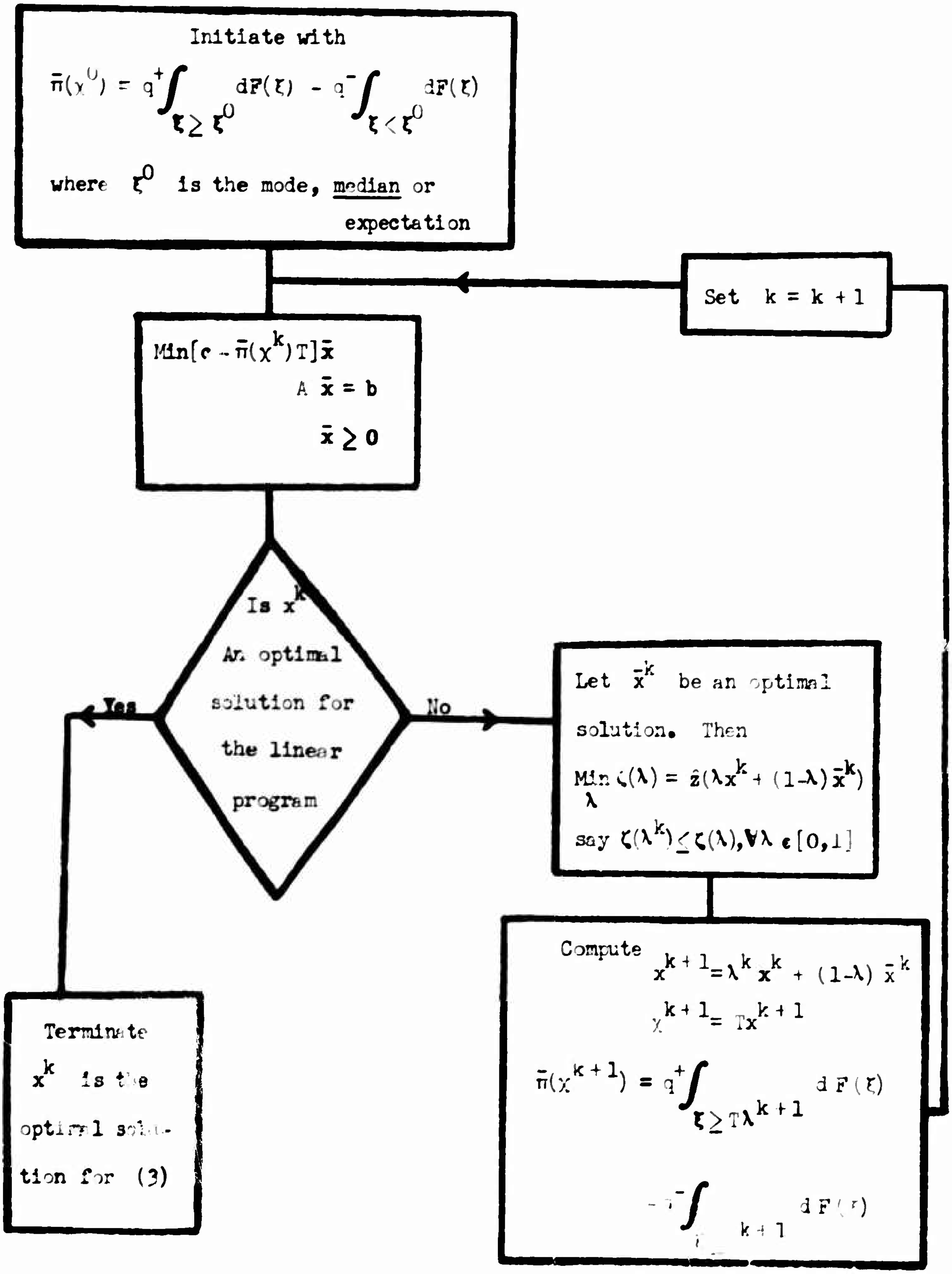
Such a M exists, because (38) is solvable $\forall k$, and $x^k(\forall^k)$ can be expressed as a convex combination of all extreme points of K. Also,

$\| [c - \pi(\chi^k)T] - [c - \pi(v^k\chi^k + (1 - v^k)\bar{\chi}^k)T] \|$ tends to zero, as $k \to \infty$.

(v^k converges to 1 and $\pi(\chi)$ is continuous). That means that at the limit we must have $\frac{D}{2} \leq 0$, which contradicts our assumption that

$[c - \pi(\chi^k)T](x^k - \bar{x}^k) > D > 0 \quad \forall k$.

Flow-Chart of the Algorithm



David Kohler wrote an experimental code for this algorithm. We used IBM 7094 and solved a few examples for which the computing time was very reasonable. An outline of this code, its features, an intuitive justification and examples are given in [8].

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APPENDIX I

We derive here an explicit expression for $Q_i(\chi_i)$ when ξ_i has no lower bound (§II.C). We recall

$$Q_i(\chi_i) = q_i^+ \bar{\xi}_i - \psi_i(\chi_i) - \pi_i(\chi_i)\chi_i$$

where

$$E\xi_i\{\xi_i\} = \bar{\xi}_i$$

$$\psi_i(\chi_i) = \tilde{q}_i \int_{\xi_i \leq \chi_i} \xi_i \, dF_i(\xi_i)$$

$$\pi_i(\chi_i) = q_i^+ - \tilde{q}_i \int_{\xi_i \leq \chi_i} dF_i(\xi_i) .$$

We divide the range of χ_i in two parts and derive explicit expressions for $Q_i(\chi_i)$ on those intervals:

Case 1. $\chi_i \leq \beta_i$ then $\{\xi_i \mid \xi_i \leq \chi_i\}$ is the set of integration for $\psi_i(\chi_i)$ and $\pi_i(\chi_i)$.

In this region, we have

$$\pi_i(\chi_i) = q_i^+ - \tilde{q}_i \int_{-\infty}^{\chi_i} dF_i(\xi_i)$$

$$\psi_i(\chi_i) = \tilde{q}_i \int_{-\infty}^{\chi_i} \xi_i \, dF_i(\xi_i)$$

$$Q_i(\chi_i) = q_i^+ \bar{\xi}_i - q_i^+ \chi_i - \tilde{q}_i \int_{-\infty}^{\chi_i} (\xi_i - \chi_i) dF_i(\xi_i) .$$

If $Q_i(\chi_i)$ is differentiable on the interval $(-\infty, \beta_i)$, we have

$$\frac{d}{dx_i} Q_i(x_i) = -q_i^+ + \tilde{q}_i \int_{-\infty}^{x_i} dF_i(\xi_i) = -\pi_i(x_i) .$$

<u>Case</u> **2.** $\beta_i < x_i$ then $\{\xi_i | \xi_i \leq x_i\} = \Xi_i$

In this region

$$\pi_i(x_i) = -q_i^-$$

$$\psi_i(x_i) = \tilde{q}_i \bar{\xi}_i$$

$$Q_i(x_i) = -q_i^- \bar{\xi}_i + q_i^- x_i$$

and

$$\frac{d}{dx_i} Q_i(x_i) = q_i^- = -\pi_i(x_i) \qquad \text{on } (\beta_i, +\infty) .$$

The function $Q_i(x_i)$ is linear on $[\beta_i, +\infty)$.

Let

$$x_i = x_{i2} + x_{i3} \text{ with } x_{i2} \leq \beta_i,\ 0 \leq x_{i3} \text{ and } T_i x - x_{i2} - x_{i3} = 0 .$$

Let

$$Q_i(x_i) = Q_i(x_{i2}, x_{i3})$$

then

$$Q_i(x_{i2}, x_{i3}) = q_i^+ \bar{\xi}_i - q_i^+ x_{i2} + \tilde{q}_i \int_{-\infty}^{x_{i2}} (x_{i2} - \xi_i) dF_i(\xi_i) + q_i^- x_{i3} .$$

In a similar manner we could have given an expression for $Q_i(x_i)$ when ξ_i has no upper bound but this could be obtained immediately from (22), by letting $\beta_i = +\infty$ and deleting the term in x_{i3}, see e.g. (29).

APPENDIX II

From (39)

$$\zeta(\lambda) = \hat{z}(\lambda x^k + (1-\lambda)\bar{x}^k)$$

$$= \lambda \sum_{j=1}^{n} c_j (x_j^k - \bar{x}_j^k) - \sum_{i=1}^{\bar{m}} \psi_i(\lambda(\chi_i^k - \bar{\chi}_i^k) + \bar{\chi}_i^k)$$

$$- \sum_{i=1}^{\bar{m}} \pi_i(\lambda(\chi_i^k - \bar{\chi}_i^k) + \bar{\chi}_i^k)(\lambda(\chi_i^k - \bar{\chi}_i^k) + \chi_i^k).$$

In order to simplify our notations, we shall write χ_i^λ for $\lambda(\chi_i^k - \bar{\chi}_i^k) + \bar{\chi}_i^k$.

Let us consider $\psi_i(\chi_i^\lambda)$ and $\pi_i(\chi_i^\lambda)$.

If $\chi_i^k - \bar{\chi}_i^k = 0$, then $\frac{d}{d\lambda}\pi_i(\chi_i^\lambda) = \frac{d}{d\lambda}\psi_i(\chi_i^\lambda) = 0$, we delete those terms.

In what follows, let us assume that $\chi_i^k - \bar{\chi}_i^k > 0$, if $\chi_i^k - \bar{\chi}_i^k \lessgtr 0$ the inequalities we obtain for the regions of λ should be reversed.

Case 1. $\chi_i^\lambda \leq \alpha_i$ then $\lambda \leq \dfrac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}$

$$\pi_i(\chi_i^\lambda) = q_i^+$$

$$\psi_i(\chi_i^\lambda) = 0$$

$$Q_i(\chi_i^\lambda) = q_i^+ \bar{\xi}_i - q_i^+ \chi_i^\lambda$$

and

$$\frac{d}{d\lambda} Q_i(\chi_i^\lambda) = -q_i^+(\chi_i^k - \bar{\chi}_i^k).$$

Case 2. $\alpha_i \leq \chi_i^\lambda \leq \beta_i$ then $\dfrac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \leq \lambda \leq \dfrac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}$

$$\pi_i(\chi_i^\lambda) = q_i^+ - \tilde{q}_i \int_{\alpha_i}^{\chi_i^\lambda} dF_i(\xi_i)$$

$$\psi_i(\chi_i^\lambda) = \tilde{q}_i \int_{\alpha_i}^{\chi_i^\lambda} \xi_i \, dF_i(\xi_i)$$

$$Q_i(\chi_i^\lambda) = q_i^+ \bar{\xi}_i - q_i^+ \chi_i^\lambda - \tilde{q}_i \int_{\alpha_i}^{\chi_i^\lambda} (\xi_i - \chi_i^\lambda) \, dF_i(\xi_i)$$

and

$$\frac{d}{d\lambda} Q_i(\chi_i^\lambda) = -q_i^+(\chi_i^k - \bar{\chi}_i^k) + \tilde{q}_i(\chi_i^k - \bar{\chi}_i^k) \int_{\alpha_i}^{\chi_i^\lambda} dF_i(\xi_i) \, .$$

<u>Case</u> 3. $\chi_i^\lambda \geq \beta_i$ then $\dfrac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \leq \lambda$

$$\pi_i(\chi_i^\lambda) = -q_i^-$$

$$\psi_i(\chi_i^\lambda) = \tilde{q}_i \, \bar{\xi}_i$$

$$Q_i(\chi_i^\lambda) = -q_i^- \bar{\xi}_i + q_i^- \chi_i^\lambda$$

and

$$\frac{d}{d\lambda} Q_i(\chi_i^\lambda) = q_i^-(\chi_i^k - \bar{\chi}_i^k) \, .$$

The points $\dfrac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}$, $\dfrac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}$ determine a change in the expressions we obtain for the derivatve of each $Q_i(\chi_i^\lambda)$ with respect to λ. The derivative of $\zeta(\lambda)$ with respect to λ will also change at those points ($2\bar{m}$ at most). But we are only interested in those which belong to $[0,1]$.

Let us define

$$\{\lambda_s\} = \{0,1, \frac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}, \frac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k}, \quad i=1,\dots,\bar{m} \mid \lambda_s < \lambda_{s+1}\} \cap [0,1]$$

and let

$$I_1^s = \left\{ i \,\middle|\, \frac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \geq \lambda_{s+1},\ \lambda_{s+1} \in \{\lambda_s\} \right\}$$

$$I_2^s = \left\{ i \,\middle|\, \frac{\alpha_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \leq \lambda_s < \lambda_{s+1} \leq \frac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k},\ \lambda_s, \lambda_{s+1}, \in \{\lambda_s\} \right\}$$

$$I_3^s = \left\{ i \,\middle|\, \frac{\beta_i - \bar{\chi}_i^k}{\chi_i^k - \bar{\chi}_i^k} \leq \lambda_s \right\} .$$

If $i \in I_1^s$, that means that on the interval $[\lambda_s, \lambda_{s+1}]$ the derivative of $Q_i(\chi_i)$ takes on the value $q_i^+(\chi_i^k - \bar{\chi}_i^k)$; if $i \in I_3^s$ we have to use the third form of the derivative of $\frac{d}{d\lambda} Q_i(\chi_i^\lambda)$. For $\lambda \in [\lambda_s, \lambda_{s+1}]$ we get:

$$\frac{d}{d\lambda} \phi(\lambda) = \sum_{j=1}^{n} c_j(x_j^k - \bar{x}_j^k) - \sum_{i \in I_1^s} q_i^+(\chi_i^k - \bar{\chi}_i^k) - \sum_{i \in I_2^s} q_i^+(\chi_i^k - \bar{\chi}_i^k)$$

$$+ \sum_{i \in I_3^s} q_i^-(\chi_i^k - \bar{\chi}_i^k) + \sum_{i \in I_2^s} \tilde{q}_i(\chi_i^k - \bar{\chi}_i^k) \int_{\alpha_i}^{\chi_i^\lambda} dF_i(\xi_i) .$$

A simple algebraic manipulation gives us (40). It is very easy to see how the construction of the sets I_1^s, I_2^s and I_3^s have to be modified if $\chi_i^k - \bar{\chi}_i^k < 0$.